

On massive cosmological scalar perturbations

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We study the cosmological perturbations of the new bi-metric gravity proposed by Hassan and Rosen [1] as a representation of massive gravity. The mass term in the model, in addition of ensuring ghost freedom for both metrics, causes the two scale factors to mix at the cosmological level and this affects the cosmological perturbation of the model. We find two combinations corresponding to the entropy and adiabatic perturbations of the theory. In this sense we show that the adiabatic perturbations could be a source for the entropy perturbations. So in addition to the adiabatic perturbations, entropy perturbations can also be present in this theory. We also show that the adiabatic perturbations are not constant at the super horizon scales, implying that the theory could not be used to describe the inflationary epoch, even if it can impose some corrections to the standard inflationary scenarios.

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I. INTRODUCTION

The current problems facing standard cosmology are of such breath and depth that any fixation to these problems requires a deeper understanding of the prevailing theory that has been and is still in use, namely the general theory of relativity. Novel ideas, both old and new, have been suggested over its rather long history in the hope of a partial remedy to some of the most pressing problems we are facing today. Two interesting observations made over the period of the past two decades, that is, the accelerated expansion of the universe and galaxy rotation curves, amongst others, have stirred a plethora of activities aimed at modifying the standard Einstein-Hilbert (EH) action and hence to offer a solution to some of these problems. With partial success, one may consider various modifications to the standard EH action in the form of modified theories of gravity with and without torsion, gravity in extra dimensions, and more recently, massive gravity, to name but a few. The latter has been attracting the attention of many experts in the field in the past few years. It is based on an old idea by Fierz and Pauli (FP) [2] where an effective field theory with a massive graviton was proposed. This theory was, however, problematic at the linearized level since the corresponding Newtonian potential was discontinuous for a vanishing mass, m^2 , resulting, for example, in a large correction to the deflection of light around the Sun compared to the experimentally accepted value [3] predicted by General Relativity (GR). This is referred to as the vDVZ discontinuity. The source of the discrepancy was traced to the degrees of freedom of the graviton, being two for a massless and five for a massive graviton. Another setback was discovered later on when it was demonstrated

that when self interacting terms are added to the action, ghosts would appear in the theory [4]. It has only been in the past few years that a method for fixing the above problems has been realized [5]. Theories rooted in the FP action are collectively known as massive gravity.

Perturbation methods are and have been an integral part of attempts to find solutions to complicated problems. Massive gravity is therefore no exception in this regard. In non-linear theories such as GR, perturbation methods can be particularly useful when one seeks the effects of small changes in the metric. Such methods have been exploited in recent years to study, for example, the structure formations in the universe. It then seems only natural to conduct such a study when dealing with massive gravity. Such a study becomes even more attractive if one considers the fact that massive gravity is inherently a bi-metric theory. This comes about since any modification to the EH action in the form of a self-coupling term involving no derivative whose definition is based on one particular metric requires an additional metric which may be dynamical or fixed [6, 7]. The appearance of a second metric in the theory can be understood on general grounds. If the second metric is non-singular, spherically symmetric, and both are diagonal in some coordinate system, a Killing horizon for one metric must be a Killing horizon for the other [8]. Interestingly, it has been shown that the off diagonal elements undergo no modification at large distances [9]. From a cosmological perspective, in a bi-metric massive gravity theory with the second metric static, there is no spatially flat FRW solutions [10], contrary to the bi-gravity formulation of the FRW cosmology for which homogeneous solutions exist [11, 12].

It has increasingly been realized that modern observational data can be explained by perturbation methods which have become an integral part of any study dealing with cosmological fluctuations. On the other hand, from a theoretical point of view, the inflationary scenario can also be supported by the modern observational data rather accurately. This makes the studies of the cosmological perturbations in massive gravity [13] all the more

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interesting since its predictions are gradually becoming known and are still not on firm grounds. As was mentioned above, one realization of massive gravity is in the language of bi-metric theories. Consequently one may be interested in a cosmological perturbation theory in such models, for the existence of two metrics in these theories may result in non-trivial features, specially in the behavior of relative metric perturbations. In this paper we will consider scalar perturbations for both metrics in a massive gravity model. We will obtain two gauge invariant combinations of the perturbed functions which can be responsible for the adiabatic and entropy perturbations. As a result we will obtain the super horizon limit of the equations of perturbations and show that the adiabatic perturbation is not constant in this scale.

The scope of the paper is as follows: in the next section we present the bi-metric model studied here. Section III deals with the cosmological perturbations of the model and definition of the gauge invariant quantities. In section IV we obtain the equations of motion for the adiabatic perturbations and discuss the super horizon limit. Conclusions are drawn in section V.

II. THE MODEL

We begin with the bimetric action [1]

$$S = -M_g^2 \int d^4x \sqrt{-g} R(g) - M_f^2 \int d^4x \sqrt{-f} R(f) + 2M_{eff}^2 m^2 \int d^4x \sqrt{-g} \sum_{n=0}^4 \beta_n e_n \left(\sqrt{g^{-1}f} \right), \quad (1)$$

where $R(g)$ and $R(f)$ are the Ricci scalars correspond to metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ respectively, and β_i are some arbitrary constants. We define the polynomials e_n as

$$\begin{aligned} e_0(\mathbb{X}) &= 1, \\ e_1(\mathbb{X}) &= [\mathbb{X}], \\ e_2(\mathbb{X}) &= \frac{1}{2} ([\mathbb{X}]^2 - [\mathbb{X}^2]), \\ e_3(\mathbb{X}) &= \frac{1}{6} ([\mathbb{X}]^3 - 3[\mathbb{X}] [\mathbb{X}^2] + 2[\mathbb{X}^3]), \\ e_4(\mathbb{X}) &= \det \mathbb{X}. \end{aligned} \quad (2)$$

with $\mathbb{X} = \sqrt{g^{-1}f}$. We also define

$$\frac{1}{M_{eff}^2} = \frac{1}{M_g^2} + \frac{1}{M_f^2}. \quad (3)$$

Without the kinetic term for metric $f_{\mu\nu}$ the action describes a theory of massive gravity which is ghost free to all orders in the decoupling limit [5]. The cosmology of such a theory was studied in [10].

The main point of the mass term is that it is symmetric

in the metrics f and g in the sense that

$$\begin{aligned} & \int d^4x \sqrt{-g} \sum_{n=0}^4 \beta_n e_n \left(\sqrt{g^{-1}f} \right) \\ &= \int d^4x \sqrt{-f} \sum_{n=0}^4 \beta_n e_{4-n} \left(\sqrt{f^{-1}g} \right). \end{aligned} \quad (4)$$

So one can read off the equation of motion for $f_{\mu\nu}$ by having the equation of motion for $g_{\mu\nu}$. The equation of motion for $g_{\mu\nu}$ can be easily obtained, resulting in [14]

$$M_g^2 G_{\mu\nu} + m^2 M_{eff}^2 J_{\mu\nu} = 0, \quad (5)$$

with

$$\begin{aligned} J_{\mu\nu} &= \beta_0 g_{\mu\nu} - \beta_1 (X_{\mu\nu} - g_{\mu\nu} e_1(\mathbb{X})) \\ &+ \beta_2 (X_{\mu\nu}^2 - X_{\mu\nu} e_1(\mathbb{X}) + g_{\mu\nu} e_2(\mathbb{X})) \\ &- \beta_3 (X_{\mu\nu}^3 - X_{\mu\nu}^2 e_1(\mathbb{X}) + X_{\mu\nu} e_2(\mathbb{X}) - g_{\mu\nu} e_3(\mathbb{X})), \end{aligned} \quad (6)$$

where $G_{\mu\nu}$ is the Einstein tensor for metric $g_{\mu\nu}$.

The equation of motion for $f_{\mu\nu}$ is

$$M_f^2 F_{\mu\nu} + m^2 M_{eff}^2 K_{\mu\nu} = 0, \quad (7)$$

with

$$\begin{aligned} K_{\mu\nu} &= \alpha_0 f_{\mu\nu} - \alpha_1 (Y_{\mu\nu} - f_{\mu\nu} e_1(\mathbb{Y})) \\ &+ \alpha_2 (Y_{\mu\nu}^2 - Y_{\mu\nu} e_1(\mathbb{Y}) + f_{\mu\nu} e_2(\mathbb{Y})) \\ &- \alpha_3 (Y_{\mu\nu}^3 - Y_{\mu\nu}^2 e_1(\mathbb{Y}) + Y_{\mu\nu} e_2(\mathbb{Y}) - f_{\mu\nu} e_3(\mathbb{Y})), \end{aligned} \quad (8)$$

where $F_{\mu\nu}$ is the Einstein tensor for metric $f_{\mu\nu}$ and we have defined $\mathbb{Y} = \sqrt{f^{-1}g}$ and $\alpha_n = \beta_{4-n}$. The cosmological solutions of this bi-metric theory has been considered in [11].

III. COSMOLOGICAL PERTURBATIONS

In order to study the cosmological perturbations of the model, we consider the scalar metric perturbations of the two metrics (our notation is compatible with that in [15])

$$\begin{aligned} ds_g^2 &= a_1^2(t) [- (1 + 2\phi_1) dt^2 + 2\partial_i B_1 dx^i dt \\ &+ [(1 - 2\psi_1)\delta_{ij} + 2\partial_i \partial_j E_1] dx^i dx^j], \end{aligned} \quad (9)$$

and

$$\begin{aligned} ds_f^2 &= a_2^2(t) [- (1 + 2\phi_2) dt^2 + 2\partial_i B_2 dx^i dt \\ &+ [(1 - 2\psi_2)\delta_{ij} + 2\partial_i \partial_j E_2] dx^i dx^j]. \end{aligned} \quad (10)$$

Let us define four zero order quantities for metrics $g_{\mu\nu}$

$$X_1 = \beta_1 a_1^2 a_2 + 2\beta_2 a_1 a_2^2 + \beta_3 a_2^3, \quad (11a)$$

$$X_2 = 2\beta_0 a_1^3 + 7\beta_1 a_1^2 a_2 + 8\beta_2 a_1 a_2^2 + 3\beta_3 a_2^3, \quad (11b)$$

$$X_3 = \beta_0 a_1^3 + 2\beta_1 a_1^2 a_2 + \beta_2 a_1 a_2^2, \quad (11c)$$

$$X_4 = \beta_0 a_1^3 + 3\beta_1 a_1^2 a_2 + 3\beta_2 a_1 a_2^2 + \beta_3 a_2^3. \quad (11d)$$

and $f_{\mu\nu}$

$$Y_1 = \beta_3 a_2^2 a_1 + 2\beta_2 a_2 a_1^2 + \beta_1 a_1^3, \quad (12a)$$

$$Y_2 = 2\beta_4 a_2^3 + 7\beta_3 a_2^2 a_1 + 8\beta_2 a_2 a_1^2 + 3\beta_1 a_1^3, \quad (12b)$$

$$Y_3 = \beta_4 a_2^3 + 2\beta_3 a_2^2 a_1 + \beta_2 a_2 a_1^2, \quad (12c)$$

$$Y_4 = \beta_4 a_2^3 + 3\beta_3 a_2^2 a_1 + 3\beta_2 a_2 a_1^2 + \beta_1 a_1^3. \quad (12d)$$

These terms can simplify the mass parts of the equations of motion. The background equations for metric $g_{\mu\nu}$ is

$$H_1^2 = \frac{1}{3} \frac{m^2 M_{eff}^2}{M_g^2} \frac{1}{a_1} X_4, \quad (13a)$$

$$H_1^2 + 2H_1' = \frac{m^2 M_{eff}^2}{M_g^2} \frac{1}{a_1} X_4, \quad (13b)$$

where $' \equiv d/dt$ and $H_1 \equiv a_1'/a_1$. Similarly for $f_{\mu\nu}$ we have

$$H_2^2 = \frac{1}{3} \frac{m^2 M_{eff}^2}{M_f^2} \frac{1}{a_2} Y_4, \quad (14a)$$

$$H_2^2 + 2H_2' = \frac{m^2 M_{eff}^2}{M_f^2} \frac{1}{a_2} Y_4 \quad (14b)$$

where $H_2 \equiv a_2'/a_2$. For the Kinetic term of the metric $g_{\mu\nu}$, we obtain, to first order [20]

$$G_{00}^{(1)} = 2\nabla^2 \left(\psi_1 + H_1(E_1' - B_1) \right) - 6H_1\psi_1', \quad (15a)$$

$$G_{0i}^{(1)} = 2\partial_i \left((\phi_1 - \frac{1}{2}B_1 H_1) H_1 - B_1 H_1' + \psi_1' \right), \quad (15b)$$

$$G_{ij}^{(1)} = \partial_i \partial_j \left(\psi_1 - \phi_1 - B_1' - 4E_1 H_1' + 2H_1 E_1' + E_1'' - 2H_1(B_1 + E_1 H_1) \right), \quad (15c)$$

and

$$\sum_{i=1}^3 G_{ii}^{(1)} = 6(\phi_1 + \psi_1) (H_1^2 + 2H_1') + 6H_1(2\psi_1' + \phi_1') + 6\psi_1'' + \nabla^2 D, \quad (15d)$$

with

$$D = 4H_1(B_1 - E_1') - 2(H_1^2 + 2H_1') E_1 - 2(\psi_1 - \phi_1 - B_1' + E_1''). \quad (16)$$

The mass term then becomes, to the first order (see the Appendix)

$$J_{00}^{(1)} = \frac{1}{a_1} \left[X_1 \left(3(\psi_2 - \psi_1) + \nabla^2(E_1 - E_2) \right) - 2X_4\phi_1 \right], \quad (17a)$$

$$J_{0i}^{(1)} = \frac{1}{2a_1} \partial_i \left[X_2 B_1 - X_1 B_2 \right], \quad (17b)$$

$$J_{ij}^{(1)} = \frac{1}{a_1} \partial_i \partial_j \left[X_2 E_1 - X_1 E_2 \right], \quad i \neq j \quad (17c)$$

and

$$\sum_{i=1}^3 J_{ii}^{(1)} = \frac{1}{a_1} \left[2X_3(\nabla^2 E_1 - 3\psi_1) + X_1(2\nabla^2 E_2 - 6\psi_2 + 3\phi_2 - 3\phi_1) \right]. \quad (17d)$$

The kinetic term for the metric $f_{\mu\nu}$ is represented by its corresponding Einstein tensor $F_{\mu\nu}$ which can be deduced for the first order perturbations by making transformation $1 \rightarrow 2$ in equations (15a)-(15d). The mass term can be obtained easily by transformations $1 \leftrightarrow 2$ and $\beta_n \rightarrow \beta_{4-n}$ in equations (17a)-(17d). The result is

$$K_{00}^{(1)} = \frac{1}{a_2} \left[Y_1 \left(3(\psi_1 - \psi_2) + \nabla^2(E_2 - E_1) \right) - 2Y_4\phi_2 \right], \quad (18a)$$

$$K_{0i}^{(1)} = \frac{1}{2a_2} \partial_i \left[Y_2 B_2 - Y_1 B_1 \right], \quad (18b)$$

$$K_{ij}^{(1)} = \frac{1}{a_2} \partial_i \partial_j \left[Y_2 E_2 - Y_1 E_1 \right], \quad i \neq j \quad (18c)$$

and

$$\sum_{i=1}^3 K_{ii}^{(1)} = \frac{1}{a_2} \left[2Y_3(\nabla^2 E_2 - 3\psi_2) + Y_1(2\nabla^2 E_1 - 6\psi_1 + 3\phi_1 - 3\phi_2) \right]. \quad (18d)$$

As we know, general relativity is invariant under coordinate transformations. Hence we restrict our model to scalar perturbations so we only consider scalar coordinate transformations as follow

$$t \rightarrow t + \delta t, \quad (19a)$$

$$x^i \rightarrow x^i + \delta^{ij} \partial_j \delta x, \quad (19b)$$

and consequently the scalar metric perturbations behave like [21]

$$\phi_i \rightarrow \phi_i - \frac{1}{a_i} (a_i \delta t)', \quad (20a)$$

$$B_i \rightarrow B_i + \delta t - \delta x', \quad (20b)$$

$$E_i \rightarrow E_i - \delta x, \quad (20c)$$

$$\psi_i \rightarrow \psi_i + H_i \delta t, \quad (20d)$$

where $i = 1, 2$. We then define six independent gauge invariant quantities as follows

$$\Psi_i = \psi_i + H_i(E_i' - B_i), \quad (21a)$$

$$\Phi_i = \phi_i - \frac{1}{a_i} (a_i(E_i' - B_i))', \quad (21b)$$

$$\mathcal{E} = E_1 - E_2, \quad (21c)$$

$$\mathcal{B} = B_1 - B_2, \quad (21d)$$

with $i = 1, 2$. Using (21c) and (21d), one can define a gauge invariant quantity

$$\Lambda = \mathcal{E}' - \mathcal{B}, \quad (22)$$

which simplifies the calculations that follow. In order to write the field equations in the gauge invariant form we can fix two scalar gauge freedoms by the following conditions

$$E_1 + E_2 = 0, \quad B_1 + B_2 = 0. \quad (23)$$

This gauge fixing has the advantage that the symmetry between the two metrics remains true, even after the gauge fixing. We then rewrite the kinetic term to first order for metric $g_{\mu\nu}$ i.e. relations (15a)-(15d) in terms of six gauge invariant quantities $\Psi_{1,2}$, $\Phi_{1,2}$, Λ and \mathcal{E} as

$$G_{00}^{(1)} = 2\nabla^2 \Psi_1 - 3H_1(2\Psi_1 - H_1\Lambda)', \quad (24a)$$

$$G_{0i}^{(1)} = \partial_i \left[2(\Psi_1' + H_1\Phi_1) + \frac{3}{2}H_1^2\Lambda - \frac{1}{2}(H_1^2 + 2H_1')\mathcal{E}' \right], \quad (24b)$$

$$G_{ij}^{(1)} = \partial_i \partial_j [\Psi_1 - \Phi_1 - (H_1^2 + 2H_1')\mathcal{E}], \quad (24c)$$

and

$$\begin{aligned} \sum_{i=1}^3 G_{ii}^{(1)} &= \nabla^2 [2(\Psi_1 - \Phi_1) - (H_1^2 + 2H_1')\mathcal{E}] \\ &\quad + 6\Psi_1'' + 6H_1(\Phi_1 + 2\Psi_1)' - 3(H_1'' + H_1H_1')\Lambda \\ &\quad + 6(H_1^2 + 2H_1')(\Phi_1 + \Psi_1), \end{aligned} \quad (24d)$$

and similarly for metric $f_{\mu\nu}$ we obtain [22]

$$F_{00}^{(1)} = 2\nabla^2 \Psi_2 - 3H_2(2\Psi_2 + H_2\Lambda)', \quad (25a)$$

$$F_{0i}^{(1)} = \partial_i \left[2(\Psi_2' + H_2\Phi_2) - \frac{3}{2}H_2^2\Lambda + \frac{1}{2}(H_2^2 + 2H_2')\mathcal{E}' \right], \quad (25b)$$

$$F_{ij}^{(1)} = \partial_i \partial_j [\Psi_2 - \Phi_2 + (H_2^2 + 2H_2')\mathcal{E}], \quad (25c)$$

and

$$\begin{aligned} \sum_{i=1}^3 F_{ii}^{(1)} &= \nabla^2 [2(\Psi_2 - \Phi_2) + (H_2^2 + 2H_2')\mathcal{E}] \\ &\quad + 6\Psi_2'' + 6H_2(\Phi_2 + 2\Psi_2)' + 3(H_2'' + H_2H_2')\Lambda \\ &\quad + 6(H_2^2 + 2H_2')(\Phi_2 + \Psi_2). \end{aligned} \quad (25d)$$

In terms of gauge invariant quantities, the mass terms (17a)-(17d) can be rewritten as

$$\begin{aligned} J_{00}^{(1)} &= \frac{1}{a_1} \left[\frac{3}{2}X_1[2(\Psi_1 + \Psi_2) + (H_2 + H_1)\Lambda] + \frac{2}{3}\nabla^2 \mathcal{E} \right. \\ &\quad \left. - X_4[2\Phi_1 + H_1\Lambda + \Lambda'] \right], \end{aligned} \quad (26a)$$

$$J_{0i}^{(1)} = \frac{X_1 + X_2}{4a_1} \partial_i (\mathcal{E}' - \Lambda), \quad (26b)$$

$$J_{ij}^{(1)} = \frac{X_1 + X_2}{2a_1} \partial_i \partial_j \mathcal{E}, \quad i \neq j \quad (26c)$$

$$\begin{aligned} \sum_{i=1}^3 J_{ii}^{(1)} &= \frac{1}{a_1} \left[(X_3 - X_1)\nabla^2 \mathcal{E} - 6(X_3\Psi_1 + X_1\Psi_2) \right. \\ &\quad \left. + 3(H_1X_3 - H_2X_1)\Lambda \right. \\ &\quad \left. + \frac{3}{2}X_1[2(\Phi_2 - \Phi_1) - (H_1 + H_2)\Lambda - 2\Lambda'] \right], \end{aligned} \quad (26d)$$

and similarly for (18a)-(18d)

$$\begin{aligned} K_{00}^{(1)} &= \frac{1}{a_2} \left[\frac{3}{2}Y_1[2(\Psi_2 + \Psi_1) - (H_2 + H_1)\Lambda - \frac{2}{3}\nabla^2 \mathcal{E}] \right. \\ &\quad \left. - Y_4[2\Phi_2 - H_2\Lambda - \Lambda'] \right], \end{aligned} \quad (27a)$$

$$K_{0i}^{(1)} = -\frac{Y_1 + Y_2}{4a_2} \partial_i (\mathcal{E}' - \Lambda), \quad (27b)$$

$$K_{ij}^{(1)} = -\frac{Y_1 + Y_2}{2a_2} \partial_i \partial_j \mathcal{E}, \quad i \neq j \quad (27c)$$

$$\begin{aligned} \sum_{i=1}^3 K_{ii}^{(1)} &= \frac{1}{a_2} \left[(Y_1 - Y_3)\nabla^2 \mathcal{E} - 6(Y_3\Psi_2) \right. \\ &\quad \left. + Y_1\Psi_1) - 3(H_2Y_3 - H_1Y_1)\Lambda \right. \\ &\quad \left. + \frac{3}{2}Y_1[2(\Phi_1 - \Phi_2) + (H_1 + H_2)\Lambda + 2\Lambda'] \right]. \end{aligned} \quad (27d)$$

By transforming to a spatial Fourier space, ∇^2 is replaced by $-k^2$, where k is the wave number. Using the above equations and after some algebra we arrive at writing \mathcal{E} and Λ as functions of Φ_i and Ψ_i and deduce the following constraints

$$\mathcal{E} = A_1(\Psi_1 - \Phi_1) = -A_2(\Psi_2 - \Phi_2), \quad (28)$$

and

$$\begin{aligned} \Lambda &= -C_1 \left[2(H_1\Phi_1 + \Psi_1') - \frac{1}{2A_1} (A_1(\Psi_1 - \Phi_1))' \right] \\ &= C_2 \left[2(H_2\Phi_2 + \Psi_2') - \frac{1}{2A_2} (A_2(\Psi_2 - \Phi_2))' \right], \end{aligned} \quad (29)$$

where

$$\frac{1}{A_1} = H_1^2 + 2H_1' - m_g \frac{X_1 + X_2}{2a_1}, \quad (30)$$

$$\frac{1}{A_2} = H_2^2 + 2H_2' - m_f \frac{Y_1 + Y_2}{2a_2}, \quad (31)$$

$$\frac{1}{C_1} = -m_g \frac{X_1 + X_2}{4a_1} + \frac{3}{2}H_1^2, \quad (32)$$

$$\frac{1}{C_2} = -m_f \frac{Y_1 + Y_2}{4a_2} + \frac{3}{2}H_2^2, \quad (33)$$

and

$$m_g = \frac{m^2 M_{eff}^2}{M_g^2}, \quad m_f = \frac{m^2 M_{eff}^2}{M_f^2}. \quad (34)$$

Using the background equations (13a), (13b), (14a) and (14b), one can simplify the quantities A_i and C_i as follows

$$A_1 = \frac{1}{2}C_1 = \frac{2a_1}{m_g}(2X_4 - X_1 - X_2)^{-1}, \quad (35)$$

$$A_2 = \frac{1}{2}C_2 = \frac{2a_2}{m_f}(2Y_4 - Y_1 - Y_2)^{-1}, \quad (36)$$

The second constraint can be simplified with the use of (28) and (35) with the result

$$-A_1(H_1\Phi_1 + \Psi'_1) = A_2(H_2\Phi_2 + \Psi'_2), \quad (37)$$

which can be used to simplify the calculations. The dynamical equations then become

$$-2k^2\Psi_1 - 3H_1(2\Psi_1 - H_1\Lambda)' = m_g \left[-\frac{3}{2a_1}X_1 \left(2(\Psi_2 - \Psi_1) + (H_1 + H_2)\Lambda - \frac{2}{3}k^2\mathcal{E} \right) + \frac{X_4}{a_1}(2\Phi_1 + H_1\Lambda + \Lambda') \right] \quad (38)$$

$$-2k^2\Psi_2 - 3H_2(2\Psi_2 + H_2\Lambda)' = m_f \left[-\frac{3}{2a_2}Y_1 \left(2(\Psi_1 - \Psi_2) - (H_1 + H_2)\Lambda + \frac{2}{3}k^2\mathcal{E} \right) + \frac{Y_4}{a_2}(2\Phi_2 - H_2\Lambda - \Lambda') \right] \quad (39)$$

and

$$\begin{aligned} & -k^2[2(\Phi_1 - \Psi_1) - (H_1^2 + 2H_1')\mathcal{E}] + 6H_1(\Phi_1 + 2\Psi_1)' + 6\Psi_1'' - 3(H_1'' + H_1H_1')\Lambda + 6(H_1^2 + 2H_1')(\Phi_1 + \Psi_1) \\ & = -\frac{m_g}{a_1} \left[-k^2(X_3 - X_1)\mathcal{E} - 6(X_3\Psi_1 + X_1\Psi_2) + 3(H_1X_3 - H_2X_1)\Lambda + \frac{3}{2}X_1[2(\Phi_2 - \Phi_1) - (H_1 + H_2)\Lambda - 2\Lambda'] \right] \end{aligned} \quad (40)$$

$$\begin{aligned} & -k^2[2(\Phi_2 - \Psi_2) + (H_2^2 + 2H_2')\mathcal{E}] + 6H_2(\Phi_2 + 2\Psi_2)' + 6\Psi_2'' + 3(H_2'' + H_2H_2')\Lambda + 6(H_2^2 + 2H_2')(\Phi_2 + \Psi_2) \\ & = -\frac{m_f}{a_2} \left[k^2(Y_3 - Y_1)\mathcal{E} - 6(Y_3\Psi_2 + Y_1\Psi_1) - 3(H_2Y_3 - H_1Y_1)\Lambda + \frac{3}{2}Y_1[2(\Phi_1 - \Phi_2) + (H_1 + H_2)\Lambda + 2\Lambda'] \right]. \end{aligned} \quad (41)$$

where we keep in mind that the quantities Λ and \mathcal{E} must be replaced by (28) and (29). In this sense we obtain four equations for four dynamical variables.

IV. EQUATIONS OF MOTION AND THE SUPER HORIZON LIMIT

From [12] we know that a classical solution for the scale factors can be deduced by assuming $a_1 = a_2$. Due to this special solution one may expect that a suitable combination of the variables $\Psi = \Psi_1 + \Psi_2$ and $\Phi = \Phi_1 + \Phi_2$ is responsible for the adiabatic perturbations. Similarly, a suitable combination of the variables $\psi = \Psi_1 - \Psi_2$ and $\phi = \Phi_1 - \Phi_2$ which is orthogonal to the adiabatic one is responsible for the entropy perturbations. To this end,

we assume $a_1 = a_2 = a$. We also define constants x_i and y_i as

$$X_i = a^3 x_i, \quad Y_i = a^3 y_i, \quad (42)$$

which, using equations (13a) and (14a), imply that

$$m_g x_4 = m_f y_4. \quad (43)$$

In order to write the equations in terms of the new variables Ψ, Φ, ψ and ϕ , we define

$$\alpha = \frac{1}{2} \left(\frac{x_4}{x_1} + \frac{y_4}{y_1} \right), \quad (44)$$

$$\beta = \frac{1}{2} \left(\frac{x_4}{x_1} - \frac{y_4}{y_1} \right). \quad (45)$$

The constraint equations (28) and (37) can then be written as

$$\alpha(\Psi - \Phi) + \beta(\psi - \phi) = 0, \quad (46)$$

$$\alpha(H\Phi + \Psi') + \beta(H\phi + \psi') = 0. \quad (47)$$

Equations (38) and (39) take the form

$$\begin{aligned} & 6H\Psi' + (2k^2 + \frac{\beta^2}{\alpha^2 - \beta^2}k^2)\Psi \\ & + (6H^2 - \frac{\beta^2}{\alpha^2 - \beta^2}k^2)\Phi + \frac{18\beta}{\alpha^2 - \beta^2}H^3\Lambda \\ & = \frac{\beta\alpha}{\alpha^2 - \beta^2}k^2(\phi - \psi) + \frac{18\beta}{\alpha^2 - \beta^2}H^2\psi, \end{aligned} \quad (48)$$

and

$$\begin{aligned} & 6H\psi' + (2k^2 - \frac{\alpha^2}{\alpha^2 - \beta^2}k^2 + \frac{18\alpha}{\alpha^2 - \beta^2}H^2)\psi \\ & + (\frac{\alpha^2}{\alpha^2 - \beta^2}k^2 + 6H^2)\phi - \frac{18\alpha}{\alpha^2 - \beta^2}H^3\Lambda \\ & = \frac{\alpha\beta}{\alpha^2 - \beta^2}k^2(\Psi - \Phi). \end{aligned} \quad (49)$$

Also one can rewrite the relation (29) as

$$\begin{aligned} \Lambda & = \frac{\alpha^2 - \beta^2}{3\alpha H^2} \left[\frac{3}{2}\psi' + \frac{1}{2}\phi' + H\psi + H\phi \right] \\ & = \frac{\beta^2 - \alpha^2}{3\beta H^2} \left[\frac{3}{2}\Psi' + \frac{1}{2}\Phi' + H\Psi + H\Phi \right]. \end{aligned} \quad (50)$$

Upon substituting Λ we obtain

$$\Psi' + \Phi' - \frac{k^2}{3H}\Phi + \left(2H - \frac{k^2}{3H}\right)\Psi = \frac{-6\beta}{\alpha^2 - \beta^2}H\psi, \quad (51)$$

$$\psi' + \phi' - \frac{k^2}{3H}\phi + \left(2H - \frac{6\alpha}{\alpha^2 - \beta^2}H - \frac{k^2}{3H}\right)\psi = 0. \quad (52)$$

Equation (51) has an important property, showing that the entropy perturbation can be a source for the adiabatic perturbation.

At this stage it is appropriate to take the following steps to make the equations less complicated and therefore easier to solve. From the first condition (46) we have

$$\alpha\Psi + \beta\psi = \alpha\Phi + \beta\phi, \quad (53)$$

and by the weighted addition of equations in (51) and (52) one can get

$$(\alpha\Psi + \beta\psi)' + (\alpha\Phi + \beta\phi)' - \frac{k^2}{3H}(\alpha\Phi + \beta\phi) \quad (54)$$

$$+ \left(2H - \frac{k^2}{3H}\right)(\alpha\Psi + \beta\psi) = 0. \quad (55)$$

Now by plugging (53) into the above equation we have

$$(\alpha\Psi + \beta\psi)' + \left(H - \frac{k^2}{3H}\right)(\alpha\Psi + \beta\psi) = 0. \quad (56)$$

The second condition (47) results in

$$(\alpha\Psi + \beta\psi)' = -H(\alpha\Phi + \beta\phi), \quad (57)$$

and reduces to

$$(\alpha\Psi + \beta\psi)' + H(\alpha\Psi + \beta\psi) = 0, \quad (58)$$

due to (53). Comparison of (56) with (58) shows

$$\frac{k^2}{3H}(\alpha\Psi + \beta\psi) = 0, \quad (59)$$

which leads to

$$\alpha\Psi = -\beta\psi, \quad (60)$$

and consequently from (53)

$$\alpha\Phi = -\beta\phi. \quad (61)$$

We note that (60) and (61) render equations (51) and (52) equivalent. So by plugging the above relations into (51) one finds

$$(\Psi + \Phi)' - \frac{k^2}{3H}(\Psi + \Phi) + 2\gamma H\Psi = 0, \quad (62)$$

with

$$\gamma = 1 - \frac{3\alpha}{\alpha^2 - \beta^2}. \quad (63)$$

The same procedure as above can be employed for the remaining equations (40) and (41). They, in conjunction with equations (60) and (61), can be used to arrive at an equation of the form

$$\Psi'' + \Phi'' + H(\Psi' + \Phi') + 2\gamma H^2(2\Psi - \Phi) = 0, \quad (64)$$

which is independent of (62). So for four variables ψ , ϕ , Ψ and Φ we have four independent equations (60), (61), (62) and (64). The two latter equations can be converted to a first order differential equation for Ψ and Φ by differentiating equation (62) and subtract the result from (64). The result is

$$\begin{aligned} & \left(\frac{k^2}{3H} + H\right)(\Psi + \Phi)' - \left(\frac{k^2}{3} + 2\gamma H^2\right)(\Psi + \Phi) \\ & - 2\gamma H\Psi' + 4\gamma H^2\Psi = 0. \end{aligned} \quad (65)$$

Equations (62) and (65) constitute our final resulting equations for Ψ and Φ . Now, combining the above equation with (62) results in

$$(\Psi + \Phi)'' - 2H(\Psi + \Phi)' + \left(\frac{k^2}{3} - 2\gamma H^2\right)(\Psi + \Phi) = 0, \quad (66)$$

which is the same as what is deduced in [17]. Since $H = -1/t$ as a consequence of the background equations, the solution of the above equation is

$$\Psi + \Phi = c_1 j_n \left(\frac{kt}{\sqrt{3}} \right) + c_2 y_n \left(\frac{kt}{\sqrt{3}} \right) \quad (67)$$

where c_1 and c_2 are integration constants, $j_n(x)$ and $y_n(x)$ are spherical Bessel functions and

$$n = \frac{-1 + \sqrt{1 + 8\gamma}}{2}.$$

Consequently, using the above solutions and either of (62) or (65), one can find $\Psi(t)$ and $\Phi(t)$ separately. This means that by employing (60) and (61) one can find $\psi(t)$ and $\phi(t)$ and eventually ψ_1 , ψ_2 , ϕ_1 and ϕ_2 as functions of time.

For super-horizon modes, i.e. $k \sim 0$, equation (66) results in

$$\Psi + \Phi = c_1 t + c_2 t^{-2}, \quad (68)$$

and by considering the fact that t is the conformal time and $t \in \{-\infty, 0\}$, the term “ $c_1 t$ ” is the damping term. The growing mode, i.e. “ $c_2 t^{-2}$ ” term, results in

$$\Psi = -c_2 \frac{1}{\gamma} t^{-2}, \quad \Phi = c_2 \frac{1 + \gamma}{\gamma} t^{-2} \quad (69)$$

This result suggests that in the theory of bi-metric gravity we have studied in this paper, the curvature perturbation is not constant at the super horizon limit which is compatible with [17]. The mass term in the theory cannot then be used as a manifestation of inflation. However, this result is acceptable because massive gravity is responsible for the late time accelerated expansion of the universe. So, if we add some sort of inflationary scenario to the theory, one may expect the above result to impose some corrections to the curvature perturbations at the super horizon limit which restricts the parameters of the theory to that of the slow roll parameters.

V. CONCLUSIONS

Modern observational data have become so accurate that meaningful comparisons with theoretical predictions are now a reality. This has led to the introduction of various calculational techniques, including metric perturbation methods, which have been playing an increasingly important role. In this paper we have employed such a method to study a bi-metric massive gravity theory. The present work proposes a full discussion of the scalar perturbations of the bi-metric gravity, with all the potential terms at our disposal. Because the mass term mixes the scale factors of the two metrics, we expect both the entropy and adiabatic perturbations to be non-zero and depend upon each other. This can also be seen from the dynamical equations of the model. As a result, similar to the double scalar field models [18], one may expect that the adiabatic perturbation should be along the path of the classical solutions in the phase-space and the entropy perturbation be orthogonal to it. Due to [12] a classical solution in the framework used here is $a_1 = a_2$. So it seems it is natural to think that sum of the two scalar perturbations corresponds to adiabatic perturbation and

the entropy perturbation corresponds to the difference of the two scalar fields. Since we have four scalar perturbations, the suitable combinations are the terms $\Psi_1 \pm \Psi_2$ and $\Phi_1 \pm \Phi_2$. From equation (51), one can see explicitly that the entropy perturbations are a source of adiabatic perturbations which is in agreement with double scalar field models [18].

Interestingly, equation (68) shows that the super horizon adiabatic perturbations are not constant for the bi-metric model discussed in this paper. We may expect that the mass term for graviton can only impose some corrections to the results of the inflationary scenario which we must add to the theory using other methods. In this scenario, the mass term in the model gets restricted by the slow roll parameters. This can be the subject of a future work.

Note added

During the completion of the present work, a study of the same subject appeared in [19].

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VI. APPENDIX

In this appendix we derive the first order approximation to matrix $\mathbb{X} = \sqrt{g^{-1}f}$. Using equations (9) and (10) we obtain, to the first order in perturbation, for matrix $\mathbb{G} = g^{-1}f$

$$\mathbb{G} = \mathbb{G}^{(0)} + \epsilon \mathbb{G}^{(1)} + \mathcal{O}(\epsilon^2), \quad (70)$$

where

$$\left(\mathbb{G}^{(0)}\right)_{\nu}^{\mu} = \left(\frac{a_2}{a_1}\right)^2 \delta_{\nu}^{\mu}, \quad (71)$$

and

$$\left(\mathbb{G}^{(1)}\right)_0^0 = -2 \left(\frac{a_2}{a_1}\right)^2 (\phi_1 - \phi_2), \quad (72)$$

$$\left(\mathbb{G}^{(1)}\right)_i^0 = \left(\frac{a_2}{a_1}\right)^2 \partial_i (B_1 - B_2), \quad (73)$$

$$\left(\mathbb{G}^{(1)}\right)_j^i = -2 \left(\frac{a_2}{a_1}\right)^2 [(\psi_1 - \psi_2) \delta_j^i + \partial_i \partial_j (E_1 - E_2)]. \quad (74)$$

We finally expand the matrix $\mathbb{X} = \sqrt{\mathbb{G}}$

$$\mathbb{X} = \mathbb{X}^{(0)} + \epsilon \mathbb{X}^{(1)} + \mathcal{O}(\epsilon^2), \quad (75)$$

where

$$\left(\mathbb{X}^{(0)}\right)_{\nu}^{\mu} = \frac{a_2}{a_1} \delta_{\nu}^{\mu}, \quad (76)$$

and

$$\left(\mathbb{X}^{(1)}\right)_{\nu}^{\mu} = \frac{a_2}{2a_1} \left(\mathbb{G}^{(1)}\right)_{\nu}^{\mu}. \quad (77)$$

The expansion of matrix $\mathbb{Y} = \sqrt{f^{-1}g}$ can be easily obtained by observing that metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ can be transformed into each other by transformation $1 \rightarrow 2$.

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- [20] Note that in [15] the Einstein tensor has been written with an upper and a lower index but we write it with both lower indices.
- [21] Note that we employ the scalar coordinate transformations for both metrics. It implicitly means that we assume both metrics live on the same manifold (this assumption may cause some ambiguities in the definition of the physical distance and parallel transportation which is out of the scope of the present paper. However we mentioned in [12] that a natural way of resolving this problem is by considering the average of both metrics as the physical metric at least in the linear regime). However Arkani-Hamed et al. [16] have assumed that each metric belongs to a certain manifold and the mass term effectively stitches these manifolds together. In this point of view each metric is invariant under its corresponding coordinate transformation.
- [22] We note that in order to read the equations of metric $f_{\mu\nu}$ from the equations of metric $g_{\mu\nu}$, one may make the transformations $\Lambda \rightarrow -\Lambda$ and $\mathcal{E} \rightarrow -\mathcal{E}$.